DETAILS IN TRIANGLES

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CHAPTER THREE

TRIANGLES

Introduction

Plane Geometry (sometimes called Euclidean Geometry) is a branch of Geometry dealing with the properties of flat surfaces and plane figures, such as triangles, quadrilaterals or circles. In this chapter give more attention to triangle.

Triangle is a shape which you should be familiar with as they are one of the most important shapes in mathematics. A triangle has three sides and three corners giving three internal angles and can be completely described by the lengths of the three sides and size of the three angles.

Objectives

At the end of this chapter, the students will be able to:

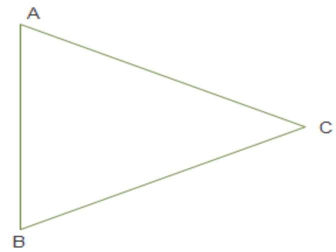
- Define triangles.
- Classifying triangles according to its side and angles.
- Find median, altitude and related lines with triangles.
- Define similar triangles.
- Prove congruence theorems of triangles.
- Identify some triangles with its applications.

Definition: A simple closed figure made of three line segments is called a **triangle**. Tri' means 'three' it indicates a triangle has three vertices, three sides and three angles. For example, in triangle ABC, denoted as ΔABC; AB, BC and CA are the three sides, $\angle A$, $\angle B$ and $\angle C$ are the three angles and point A, point B and point C are three vertices.

1.1 Definitions of triangles

Definition: Side of a triangle is said to be opposite to an angle of a triangle if and only if the side does not contain the vertex of the angle. The angle is also said to be opposite to the side.

Example: Let us seeΔABCin the figure given below



- ∠A is opposite to side BC or side BC is opposite to ∠A
- ∠B is opposite to side AC or side AC is opposite to ∠B
- ∠C is opposite to side AB or side AB is opposite to ∠C

Definition: Side of a triangle is said to be included between two angles if and only if the vertices of the angles are end points of the side. An angle of a triangle is said to be included between two sides of the triangle if and only if the sides of the angle contain the two sides of the triangle.

From the above figure (ΔABC);

- ∠A is included angle between sides AB and AC; side AB is included between ∠A and ∠B
- ∠B is included angle between sides BA and BC; side AC is included between ∠A and ∠C
- ∠C is included angle between sides CA and CB; side BC is included between ∠C and ∠B

3.2 Classifications of Triangles

Classify triangles based on their sides and their angles.

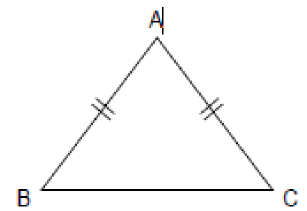
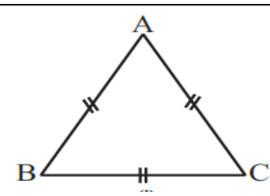
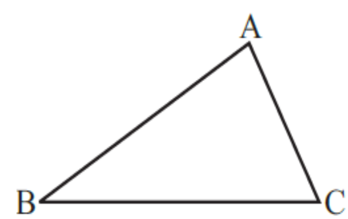
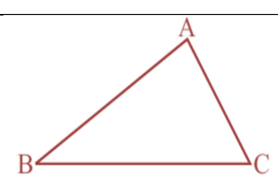
Based on Their Sides:

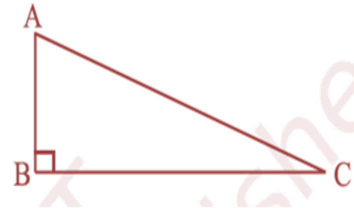
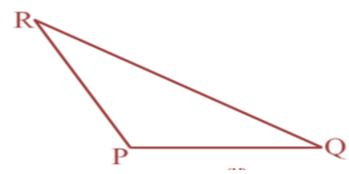
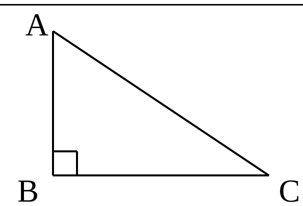
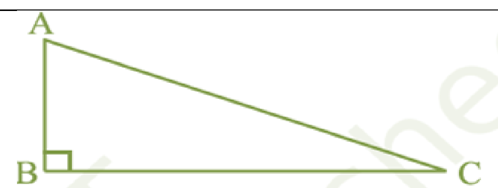
- Scalene triangle
- Isosceles triangle and
- Equilateral triangles.

Based on Their Angles:

- Acute-angled triangle
- Obtuse-angled triangle and
- Right-angled triangle.

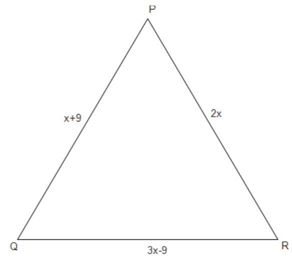
The following table summarizes basic types of triangle and its properties;

<table>
<tr><td colspan="3">Based on their sides</td></tr>
<tr><td>1</td><td>Isosceles Triangle.
➢ Any two of its sides are the same in length
➢ Base angles, opposite to the equal sides are equal in measure. The third angle is vertex angle of a triangle
➢ The third side, opposite to the vertex angle of an isosceles triangle is called its base.</td><td>
AB = AC
m(<ABC) = m(<ACB)</td></tr>
<tr><td>2</td><td>Equilateral triangles
Equipments “equal” and lateral means “sides.”
A triangle in which
➢ All sides have the same in length.
➢ Each angle has measure 60°(Equiangular)
➢ Equilateral triangle is a special Isosceles triangle.</td><td>
$\overline{AB} = \overline{BC} = \overline{AC}$
$m(< ABC) = m(< ACB) = m(< CAB)$</td></tr>
<tr><td>3</td><td>Scalene Triangles
All the three sides are not equal and the three angles are not the same measure.</td><td>
$\overline{AB} \neq \overline{BC} \neq \overline{AC}$
$m(< ABC) \neq m(< ACB) \neq m(< CAB)$</td></tr>
<tr><td colspan="3">Based on their Angles</td></tr>
<tr><td>1</td><td>Acute – Angled Triangle
➢ A triangle with all the three interior angles measure less than 90^0. Or
➢ A triangle with 3 acute angles.
Acute Isosceles Triangle: if the length of the base is shorter than two times the length of the altitude.</td><td>
$m(<A) < 90^0$, $m(<B) < 90^0$and $m(<C) < 90^0$</td></tr>
</table>

2	**Right – Angled Triangle.** ➢ A triangle with one of its interior angle measure is equal to 90^0. ➢ The side opposite to the right angles called the **hypotenuse**; the other two sides are known as the **legs** of the right-angled triangle. ➢ The square on the hypotenuse = sum of the squares on the two legs. ($AB^2 + BC^2 = AC^2$)	 $m(<ABC) = 90^0$ $\overline{AC}$is Hypotenuse, $\overline{AB}$and $\overline{BC}$ are legs of ΔABC
3	**Obtuse – Angled Triangle.** ➢ A triangle with one of the interior angle measure is greater than 90° Obtuse Isosceles Triangle: if the length of the base is longer than two times the length of the altitude.	 $M(<RPQ) > 90^0$
	Combining Their Names	
1	**Right – Isosceles Triangle.** ➢ Two legs are equal in length. ➢ One of its interior angle is right angle.	 $\overline{AB} \equiv \overline{BC}$ and $m(<ABC) = 90^0$ $\overline{AB}$ is Hypotnuse side
2	**Right – Scalene Triangle.** ➢ All the three sides are not equal in length. ➢ One of its interior angle measure is right angle(90^0)	 $\overline{AB} \neq \overline{BC} \neq \overline{AC}$ and $m(<ABC) = 90^0$

Activity 1

1. Find x and the measure of each side of an equilateral triangle PQR.



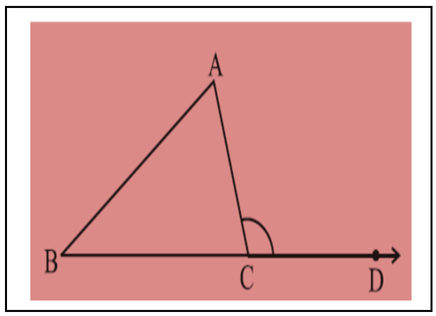
Note:

- When we say angles ofΔABC we mean that the interior angles of ΔABC.
- In any triangle; the largest angle is opposite to the longest side; the smallest angle is opposite to the shortest side; the remaining angle is opposite the remaining side.
- The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.

A triangle has essentially the following seven elements

- The three sides
- The three angles and
- One Area.

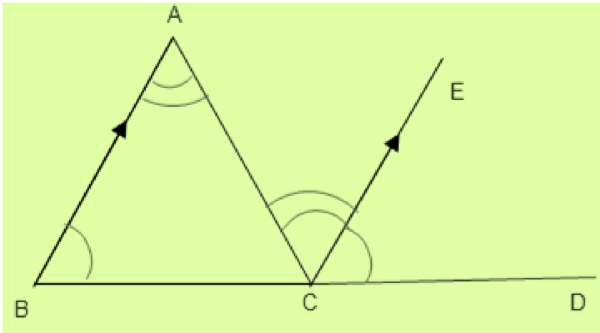
Given ΔABC and extend one of its sides, say BC to$\overrightarrow{BD}$.



Produce $\angle$ACD at point C. This angle lies in the exterior of ΔABC.
We call it an **exterior angle** of ΔABC formed at vertex C.
Clearly $\angle$BCA is an adjacent angle to $\angle$ACD. The remaining two angles of ΔABC namely $\angle$A and $\angle$B are called the two **interior opposite angles (**remote interior angles)of $\angle$ACD.
At each vertex, you have two ways of forming an **exterior angle**.
An exterior angle of a triangle is equal to the sum of its interior opposite angles.

i.e. $m(\angle ACD) = m(\angle A) + m(\angle B)$

Explain a proof that the exterior angle of a triangle equals the sum of the two interior opposite angles, using the construction below.



Proof: construct a line through vertex C parallel to $\overline{AB}$

i. m(∠A) = m(∠ACE), Alternate interior angles as $\overline{AB} \parallel \overline{CE}$

ii. m(∠B) = m(∠ECD), Corresponding angles as $\overline{AB} \parallel \overline{CE}$

m(∠ACD) = m(∠ACE) + m(∠ECD) = m(∠A) + m(∠B) from (i) and (ii)

3.3. Congruence of Triangles

You must have observed that two copies of your photographs of the same size are identical. Similarly, two bangles of the same size, two ATM cards issued by the same bank are identical such figures are called **congruent figures** (‘congruent’ means equal in all respects or figures whose shapes and sizes are both the same). Now, draw two circles of the same radius and place one on the other, they cover each other completely and we call them as congruent circles. You may wonder why we are studying congruence. So, you can find numerous examples where congruence of objects is applied in daily life situations.

Roughly speaking two geometrical figures are said to be congruent if they have the same shape and size.

Note:

- Two line segments $\overline{AB}$ and $\overline{CD}$ are said to be congruent; written as $\overline{AB} \equiv \overline{CD}$ if and only if they have the same length.

  Symbolically;

$$\overline{AB} \equiv \overline{CD} \Leftrightarrow AB = CD$$

- Two angles ∠ABC and ∠DEF are said to be congruent; written as ∠ABC ≡ ∠DEF if and only if they have the same measure.

  Symbolically;

$$\angle ABC \equiv \angle DEF \Leftrightarrow m(\angle ABC) = m(\angle DEF)$$

Definition:

Two triangles ΔABC and ΔDEF are said to be congruent written$\Delta ABC \equiv \Delta DEF$, if and only if their corresponding sides are congruent and their corresponding angles are congruent.

Symbolically;

$\Delta ABC \equiv \Delta DEF$, if and only if

i. Their corresponding angles are congruent

   i.e: $\angle A \equiv \angle D, \angle B \equiv \angle E$ and $\angle C \equiv \angle F$

ii. Their corresponding sides are congruent.

   i.e:$\overline{AB} \equiv \overline{CD}, \overline{BC} \equiv \overline{DE}$ and $\overline{AC} \equiv \overline{EF}$

Exercise

Show that the following statements are congruent

a. For any triangle ABC, $\Delta ABC \equiv \Delta ABC$
b. For all triangles ABC and DEF, if $\Delta ABC \equiv \Delta DEF$, then $\Delta DEF \equiv \Delta ABC$
c. For all triangles ABC, DEF and PQR, if $\Delta ABC \equiv \Delta DEF$, $\Delta DEF \equiv \Delta PQR$ then $\Delta ABC \equiv \Delta PQR$

Note: when drawing congruent triangles, you can mark the corresponding parts in the same way to easily see corresponding congruent parts.

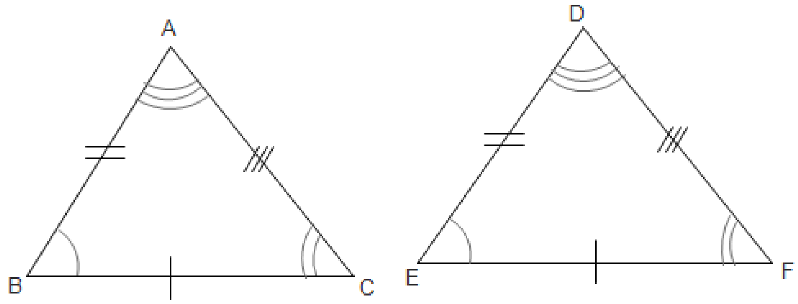
Definition: If the six parts of one triangle are congruent to the corresponding six parts of another triangle, then the triangles are congruent triangles (Two triangles are congruent if and only if their corresponding parts are congruent).

Congruent triangles are triangles that have the same

- Size and the same shape.
- Exact duplicates of each other.
- Such triangles can be moved on top of one another so that their corresponding parts line up exactly.

Given ΔABC and ΔDEF in the figure below;

What are the congruent parts in ΔABC andΔDEF?



$\Delta ABC$is congruent to ΔDEF denoted by $\Delta ABC \equiv \Delta DEF$, if and only if the following conditions are always true.

Corresponding sides are congruent Corresponding angles are congruent

$\overline{AB} \equiv \overline{DE}\angle A \equiv\angle D$

$\overline{BC} \equiv \overline{EF}\angle B \equiv\angle E$

$\overline{AC} \equiv \overline{DF}\angle C \equiv\angle F$

If $\Delta ABC \equiv \Delta DEF$ then the following statements also holds true;

$$\Delta ACB \equiv \Delta DFE \Delta BAC \equiv \Delta EDF$$

$$\Delta BCA \equiv \Delta EFD \Delta CBA \equiv \Delta FED$$

$$\Delta CAB \equiv \Delta FDE$$

3.4 Theorems on Congruency of Triangles

As seen in the above section; two triangles are congruent when all of the six corresponding parts are congruent. However, you do not need to know all of the six corresponding parts to conclude that the triangles are congruent. Each of the following Theorems states that three corresponding parts determine the congruence of two triangles.

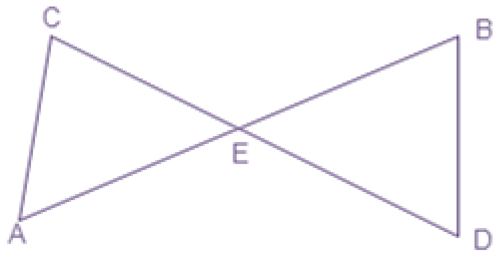
Theorem 3.4.1: The two Sides and included Angle Postulate (SAS)

Two sides and the included angle of one triangle must be congruent to two sides and the included angle of the other triangle. Then the two triangles are congruent by SAS

This result cannot be proved with the help of previously known results and so it is accepted true as an axiom

Symbolically, Given ΔABCandΔDEF, if $\overline{AB} \equiv \overline{DE}$, $\angle B\equiv\angle E$ and $\overline{BC} \equiv \overline{EF}$ then $\Delta ABC \equiv \Delta DEF$

Example: In the figure below $\overline{AB}$ and $\overline{CD}$ bisect each other at point E, prove that $\Delta AEC \equiv \Delta BED$



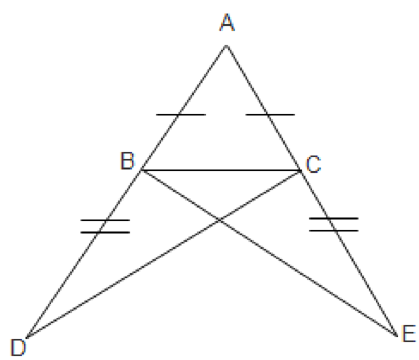
Proof:

1. $\overline{AB}$ and $\overline{CD}$ bisect each other at E ------------------------Given
2. $\overline{AE} \equiv \overline{BE}$ and $\overline{CE} \equiv \overline{DE}$ ------------------------------------- Definition of segment bisector
3. $\angle AEC \equiv \angle BED$ --- Vertically opposite angles
4. $\Delta AEC \equiv \Delta BED$ --- Step 2, 3 and SAS

Activity: in the figure given below, prove that;

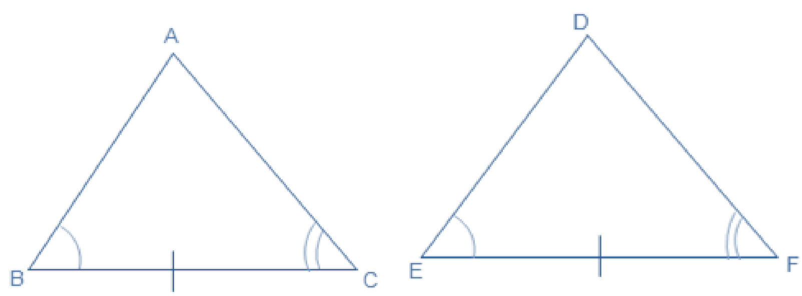
a) $\Delta DAC \equiv \Delta EAB$

b) $\overline{CD} \equiv \overline{BE}$



Theorem 3.4.2:**The two Angles and included Side Theorem (ASA)**

If two angles and the included side of one triangle are congruent to the two angles and the included side of other triangle, then the two triangles are congruent



Proof: Given two triangles ΔABC and ΔDEF in which: m(∠B) = m(∠E), m(∠C) = m(∠F) and $\overline{BC} \equiv \overline{EF}$

we need to prove that ΔABC ≡ ΔDEF.

Case I: Let AB = DE

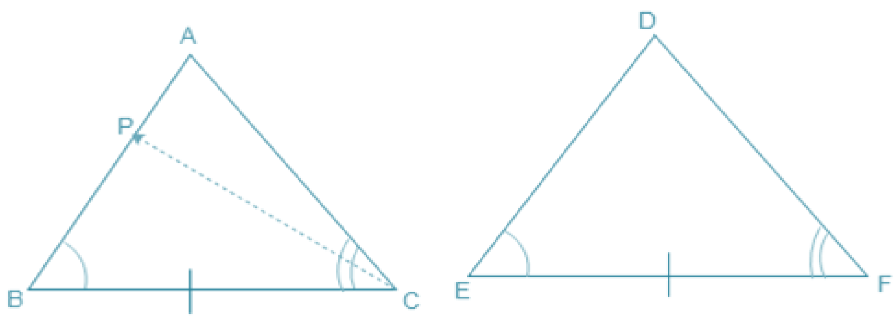
Now you may observe that

Statement	Reason
i. $\overline{AB} \equiv \overline{DE}$	Assumed
ii. m(∠B) = m(∠E)	Given
iii. $\overline{BC} \equiv \overline{EF}$	Given

So, we can conclude that:

ΔABC ≡ ΔDEF, by the SAS axiom for congruence.

Case II: Let AB > DE. So, we can take a point P on AB such that PB = DE. Now consider ΔPBC and ΔDEF



Statement	Reason
i. $\overline{PB} \equiv \overline{DE}$	Construction
ii. ∠B = ∠E	Given
iii. $\overline{BC} \equiv \overline{EF}$	Given

So, we can conclude that:

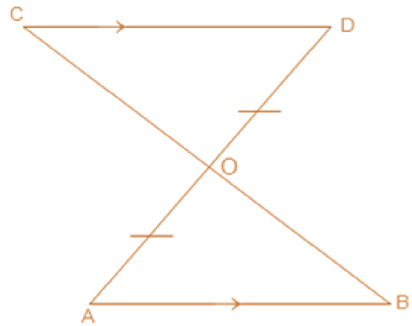
ΔPBC ≡ ΔDEF, by the SAS axiom for congruence.

Since the triangles are congruent, their corresponding parts will be equal. So, ∠PCB ≡∠DFE But, given that ∠ACB≡∠DFE. So, by transitive property of congruence∠ACB ≡ ∠PCB,this is possible only if P coincides with A or BP = BASo, ΔABC ≡ ΔDEF, by SAS axiom.

Example: Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD see in the figure below. Show that

i. ΔAOB≡ ΔDOC

ii. Point O is also the mid-point of BC.



Solution:

Consider ΔAOB and ΔDOC.

Statement Reason

i. ∠BAO ≡∠CDO ----------------------- Alternate interior angles as AB || CD and BC is Transversal

ii. ∠AOB ≡∠DOC---------------------Vertically opposite angles

$\overline{OA} \equiv \overline{OD}$ ------------------------------ Given

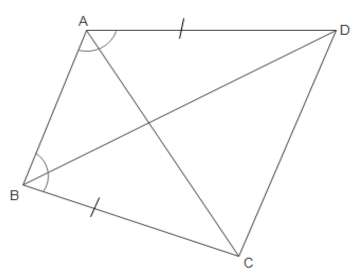
∴ ΔAOB ≡ ΔDOC ASA Theorem

b. $\overline{OB} \equiv \overline{OC}$ -----------------------------------corresponding sides of ≡ triangles.

So, O is the mid-point of BC.

Example: ABCD is a quadrilateral in which AD = BC and ∠DAB ≡ ∠CBA. Prove that

a. ΔABD ≡ ΔBAC
b. $\overline{BD} \equiv \overline{AC}$
c. ∠ABD ≡∠BAC.



Solution: Consider ΔABD and ΔBAC.

Statement Reason

i. . $\overline{AD} \equiv \overline{BC}$ --- Given

ii. ∠BAD ≡∠ABC--Given.

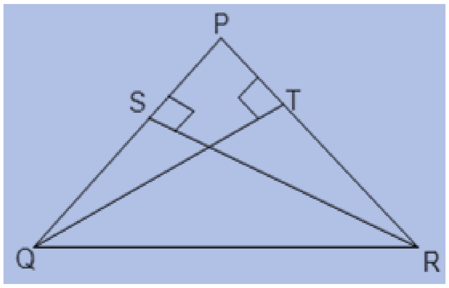
iii. $\overline{AB} \equiv \overline{BA}$ -- Common side

$\therefore \Delta ABD \equiv \Delta BAC$. SAS Theorem

iv. $\overline{BD} \equiv \overline{AC}$ --- Corresponding sides of congruent triangle.

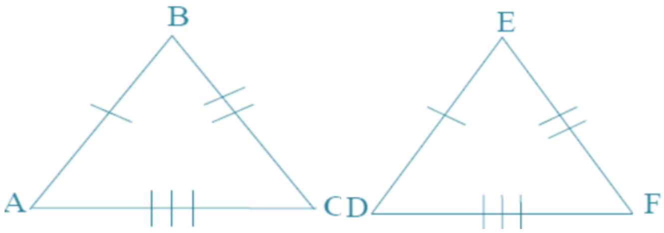
v. $\angle ABD \equiv \angle BAC$ ---------------------------------- Corresponding angles of congruent triangle.

Activity: In the figure below; $\overline{RS} \equiv \overline{QT}$ then prove that $\overline{PQ} \equiv \overline{PR}$



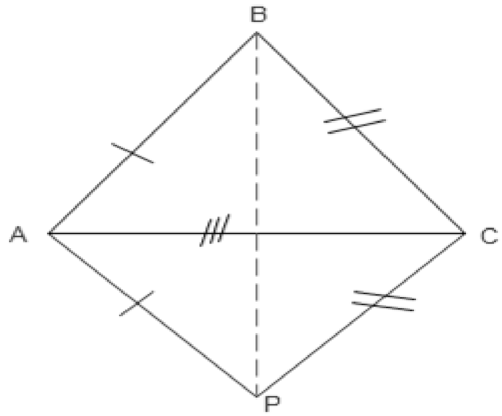
Theorem 3.3.3.Side – Side – Side Congruency Theorem (SSS)

If three sides of one triangle are congruent to the three sides of another triangle, then the two triangles are congruent.



Proof;

1. construct a line segment AP such that $\overline{AP} \equiv \overline{DE}$and $\angle EDF \equiv \angle PAC$



Statement		Reason
i.	$\overline{AP} \equiv \overline{DE}$	Construction
ii.	$\angle EDF \equiv \angle PAC$	Construction
iii.	$\overline{AC} \equiv \overline{DF}$	Given

iv. $\Delta APC \equiv \Delta DEF$. by SAS Theorem

v. $\overline{CP} \equiv \overline{FE}$ --------------------- Corresponding sides of ≡ triangles

vi. $\angle CBP \equiv \angle CPB$ ---------------- Base angle of Isosceles triangleΔCBP.

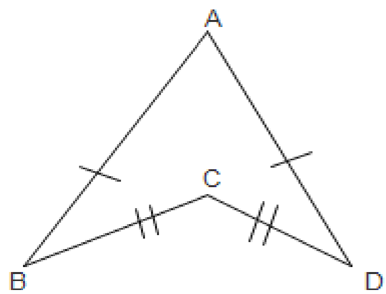
vii. $\angle ABP \equiv \angle APB$ ---------------- Base angle of Isosceles triangleΔABP.

viii. $\angle ABC \equiv \angle APC$ ---------------congruent angles sums are congruent

ix. $\Delta ABC \equiv \Delta APC$. by SAS Theorem.

x. $\Delta ABC \equiv \Delta DEF$. By steps iv, ix and transitivity Property of congruent triangles. So the theorem is proved.

Example: In the figure given below, prove that $\angle B \equiv \angle D$



Proof: Construct a line segment from point A to point C. Now, compare ΔABC andΔADC

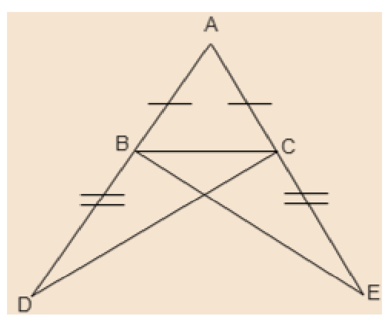
Statement Reason

1. $\overline{AB} \equiv \overline{AD}$ -- Given
2. $\overline{BC} \equiv \overline{DC}$ -- Given
3. $\overline{AC} \equiv \overline{AC}$ -- common side
4. $\Delta ABC \equiv \Delta ADC$. By steps1, 2, 3 and SSS congruency theorem
5. $\angle B \equiv \angle D$ Corresponding angles of congruent triangles.

Activity: Prove the following statements by using the figure given below.

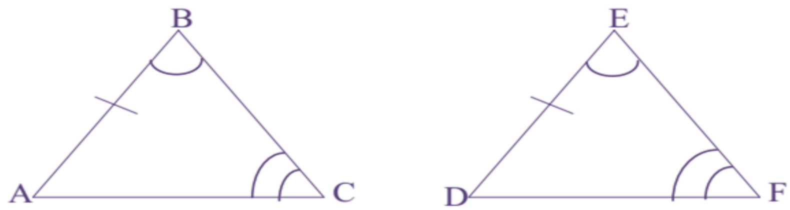
a) $\Delta BDC \equiv \Delta CEB$

b) $\angle DBC \equiv \angle ECB$



Theorem 3.3.4:Angle-Angle-Side, Congruency Theorem (AAS)

If two angles and a non-included side of one triangle are congruent to the corresponding two angles and a non-included side of a second triangle, then the two triangles are congruent.



Proof: Co4siderΔABC andΔDEF

i. $\angle A \equiv \angle D$ ------------------------ interior angle sum theorem of a triangle.

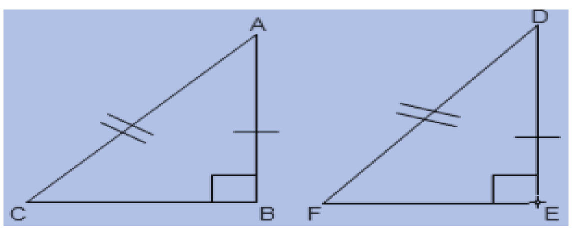
ii. $\angle B \equiv \angle E$ ------------------------ Given

iii. $\overline{AB} \equiv \overline{DE}$ ----------------------- Given

$\therefore \Delta ABC \equiv \Delta DEF$. ByASA Theorem

Theorem 3.4.5:Right angle- hypotenuse – Side theorem (RHS)

If the hypotenuse and one side of a right triangle are congruent to the hypotenuse and the corresponding side of the other right triangle, then the two right angle triangles are congruent.



Given: $\overline{AC} \equiv \overline{DF}$, $\overline{AB} \equiv \overline{DE}$, and $m(\angle B) = m(\angle E) = 90^0$. Prove that$\Delta ABC \equiv \Delta DEF$.

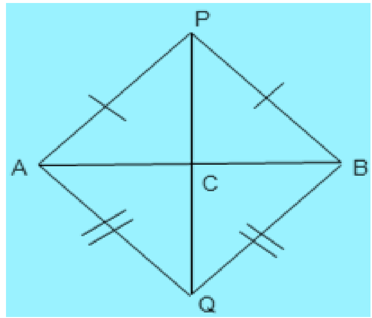
Proof:$\overline{AB} \equiv \overline{DE}$, Given

$\angle B = \angle E$, both are right angle

$\overline{BC} \equiv \overline{EF}$; Using by Pythagoras theorem

$\therefore \Delta ABC \equiv \Delta DEF$; By SAS Theorem, The theorem is proved

Example:$\overline{AB}$ is a line-segment. P and Q are points on opposite sides of $\overline{AB}$ such that each of themis equidistant from the points A and B see in the figure below. Show that the line PQ is the perpendicular bisector of AB.



Solution: Given that PA = PB and QA = QB. We need to show that;

a. PQ ⊥AB and
b. PQ bisects AB.

Let PQ intersect AB at C.

Consider ΔPAQ ≡ ΔPBQ

i. $\overline{AP} \equiv \overline{BP}$ ------------------------------ Given
ii. $\overline{AQ} \equiv \overline{BQ}$ ----------------------------- Given
iii. $\overline{PQ} \equiv \overline{PQ}$ ----------------------------- Common side
iv. ΔPAQ ≡ ΔPBQ, By SSS congruency theorem

Therefore, ∠APQ ≡∠BPQ Corresponding angles of congruent triangles.

Now let us consider Δ PAC and Δ PBC.

i. $\overline{AP} \equiv \overline{BP}$ ------------------------------ Given
ii. ∠APC≡∠BPC (∠APQ = ∠BPQ proved above in step vi)
iii. $\overline{PC} \equiv \overline{PC}$ ------------------------------- Common side

So, ΔPAC ≡ ΔPBC, By SAS congruency theorem

Therefore, $\overline{AC} \equiv \overline{BC}$. Corresponding sides of congruent triangles ……………… (1)

∠ACP ≡∠BCP. Corresponding angles of congruent triangles

Also,

m(∠ACP) + m(∠BCP) = 180° (straight angle)

2(∠ACP) = 180°

m(∠ACP) = 90° ……………………………………………………(2)

∠ACP is right angle

i.e. PQ ⊥ AB

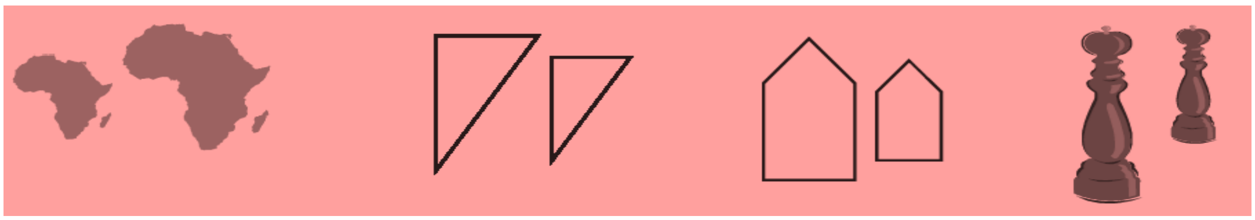
From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

3.5. Similarity of Triangles

The concept of similar triangles forms an important foundation for trigonometry, but it also can be used to solve geometric problems when a trigonometric method may be very difficult or impossible to use.

After an architect finishes the plan of a building, it is usual to prepare a model of the building. In different areas of engineering, it is usual to produce models of industrial products before moving to the actual production. What relationships do you see between the model and the actual product?

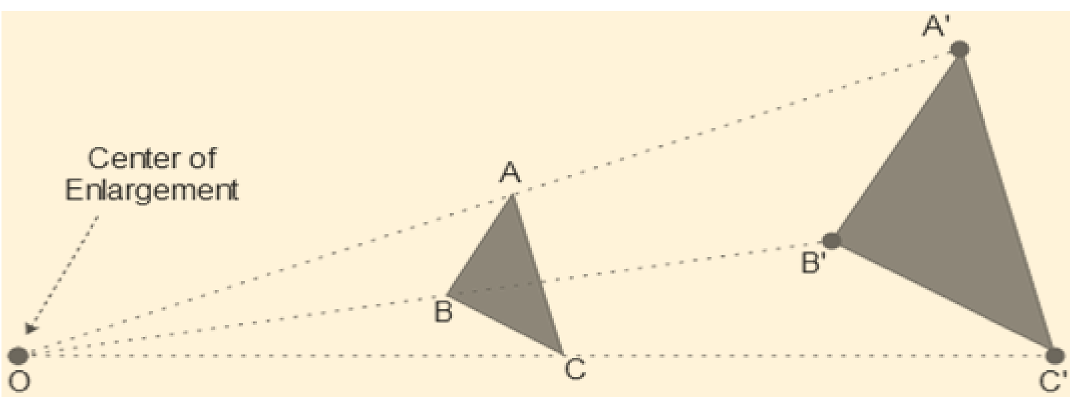
Each of the following pairs of figures are similar, with one shape being an enlargement of the other.



An overhead projector forms an image on the screen which has the same shape as the image on the transparency but with the size may differ. Figures that have the same shape but that might have different sizes are called **similar**.

An enlargement is a transformation of a plane figure in which each of the points such as A, B, C is mapped onto A', B', C' by the same scale factor, *k*, from a fixed point O. The distances of A', B', C' from the point O are found by multiplying each of the distances of A, B, C from O by the scale factor *k*.

$$OA' = k \times OA, \quad OB' = k \times OB, \quad OC' = k \times OC$$



If the scale factor of enlargement is greater than 1, then the image is larger than the object. If the scale factor lies between 0 and 1 then the resulting image is smaller than the object. The transformation is still known as an enlargement.

In two polygons of the same number of sides are similar, if

- One is an enlargement of the other.
- Angles in corresponding positions are congruent.
- Corresponding sides have the same ratio.

3.5.1. Definition of Similar triangles

In the case of triangles,

Two triangles are said to be similar if and only if;

- They will have congruent corresponding angles and
- Their corresponding sides will be in the same proportion.

Symbolically: Given ΔABC and ΔDEF are similar, written ΔABC ~ΔDEF if and only if

i. $\angle A \equiv \angle D$, $\angle B \equiv \angle E$, and $\angle C \equiv \angle F$

ii. $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = k$ is called the **scaled-factor.**

Remark:

- Congruent triangles are always similar with a scale factor equal to 1 but the converse may not true.
- If two shapes are similar, one can have considered to be an enlargement of the other.
- All circles are similar, all line segments are similar, all equilateral triangles are similar, all squares are similar, etc

3.5.2. Theorem on Similarity of Triangles

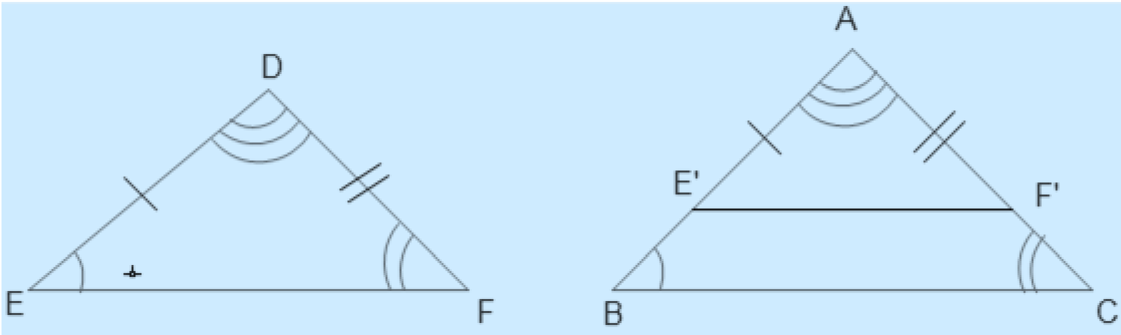
The following theorems on similarity of triangles will serve as tests to check whether or not two triangles are similar.

Theorem3.5.1. Angle- Angle – Angle similarity theorem (AAA)

If the corresponding angles in two triangles are congruent, then the triangles are similar.

Given: $\angle A \equiv \angle D$, $\angle B \equiv \angle E$ and $\angle C \equiv \angle F$

Prove: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$



i. Locate E’ on AB so that AE’ $\equiv$ DEConstruction

ii. $\angle$A$\equiv\angle$D, $\angle$B $\equiv\angle$E, $\angle$C $\equiv\angle$FGiven

iii. Locate F’ on AC so that AF’ $\equiv$ DFConstruction

iv. ΔAE’F’ $\equiv$ ΔDEFBy SAS.

v. AE’ $\equiv$ DE, E’F’ $\equiv$ EF and AF’ $\equiv$ DF Corresponding sides of $\equiv\Delta$

vi. $\angle$AE’F’ $\equiv$ $\angle$ECorresponding angles of $\equiv\Delta$

vii. $\angle$AE’F’ $\equiv$ $\angle$B Step ii, step vi and Transitivity property

viii. E’F’ $\parallel$ BC Step vii(congruent Corresponding angles)

ix. $AB/AE' = \frac{AC}{AF}$’Side Splitter Theorem

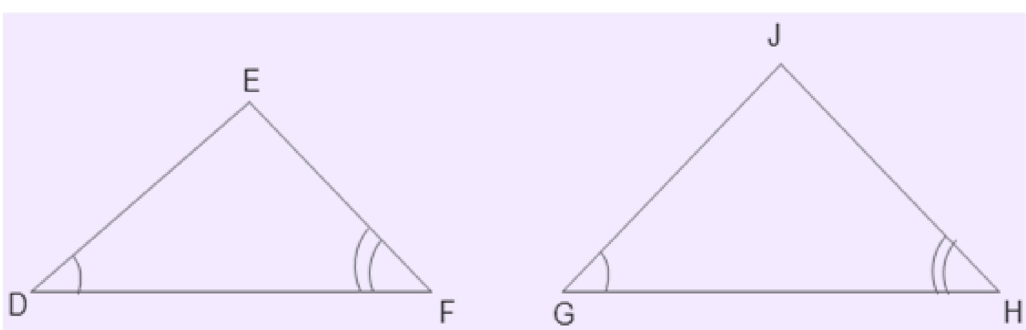
x. $AB/DE = AC/DF$, since AE’ $\equiv$ DE and AF’ $\equiv$ DF. Similarly;

xi. $AB/DE = BC/EF$

Therefore $AB/DE = BC/EF = AC/DF$ the theorem is proved.

Theorem3.5.2. Angle-Angle (AA) Similarity Theorems

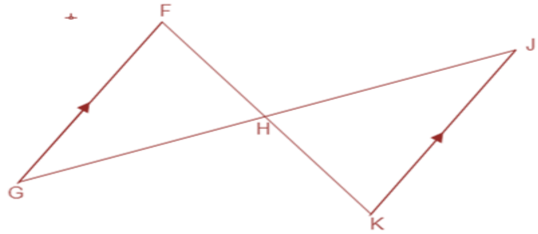
Two angles of one triangle are congruent to the corresponding two angles of another triangle then the two triangles are similar.



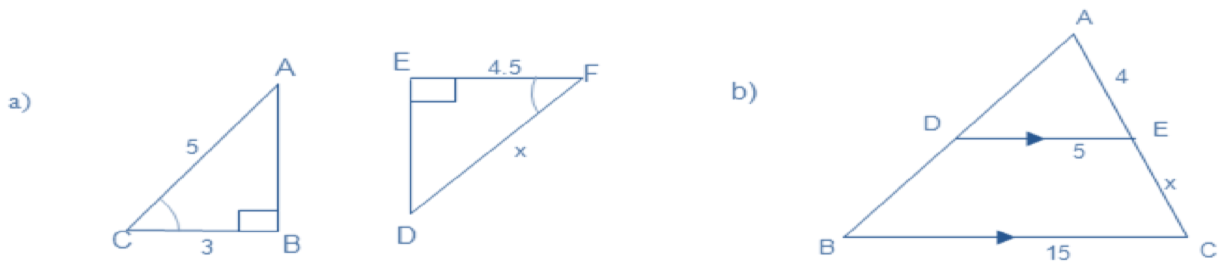
$\angle$G $\equiv\angle$D and $\angle$H $\equiv\angle$F, then ΔDEF$\sim\Delta$GJH,

Proof: left for the reader

Activity: Are the following pair of triangles similar? If so, what similarity statement can be made? Name the postulate or theorem you used.



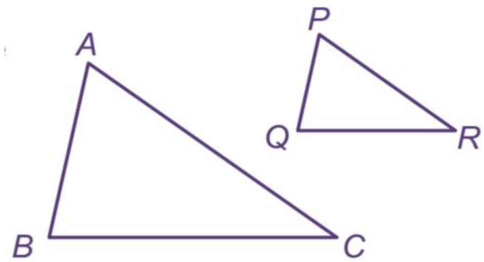
Activity: Explain why these triangles are similar. Then find the value of x.

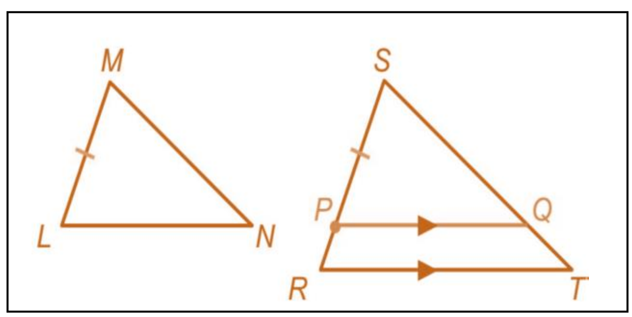


Theorem 3.5.3. Side-Side-Side Similarity Theorem (SSS)

If the lengths of the three corresponding sides of two triangles are proportional, then the triangles must be similar, that is

$$If\ \ \frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{BC}}{\overline{QR}} = \frac{\overline{AC}}{PR}\ then\ \ \Delta ABC = \Delta PQR$$





Proof:

Given: $\frac{\text{RS}}{\text{LM}} = \frac{\text{ST}}{\text{MN}} = \frac{\text{RT}}{\text{LN}}$

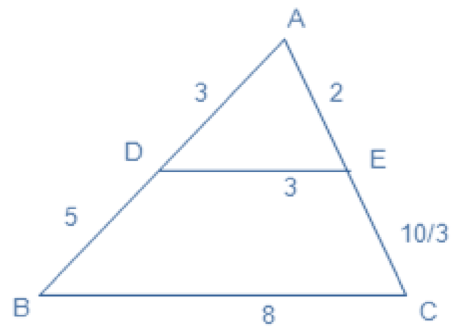
We need to show: ΔLMN ~ ΔRST Locate **P** on **RS** so that **PS** ≡**LM**. Draw **PQ** so that

PQ // RT. Then Δ**RST** ~ Δ**PSQ**, by AA Similarity theorem, then $\frac{RS}{PS} = \frac{ST}{SQ} = \frac{RT}{PQ}$ but $\frac{RS}{LM} = \frac{ST}{MN} = \frac{RT}{LN}$ is given, then

Since $\overline{\mathbf{PS}} \equiv \overline{\mathbf{LM}}$, you substitute in the given proportion and find that $\overline{\mathbf{SQ}} \equiv \overline{\mathbf{MN}}$ and $\overline{\mathbf{QP}} \equiv \overline{\mathbf{NL}}$.

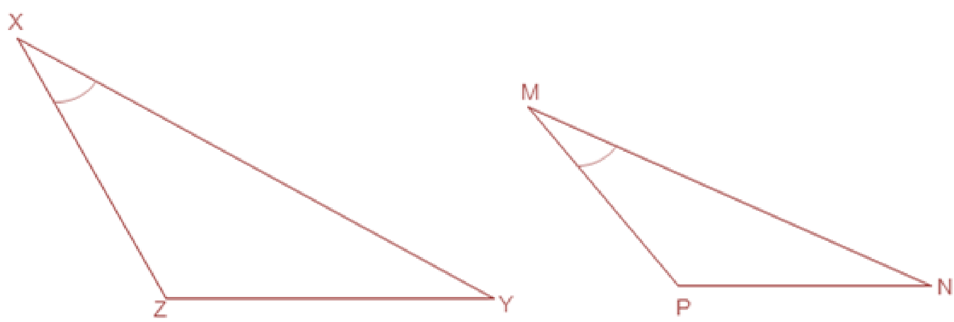
By the SSS Congruence Theorem, it follows that Δ**PSQ** ≡Δ**LMN**. Use the definition of congruent triangles and the AA Similarity Theorem to conclude that Δ**RST** ~ Δ**LMN**.

Exercise: Are the following pairs of triangles similar? If so, what similarity statement can be made? Name the postulate or theorem you used.



Theorem 3.5.4.Side-Angle-Side Similarity Theorem (SAS)

If an angle of one triangle is congruent to an angle of a second triangle and the lengths of the sides including these angles are proportional, then the triangles must be similar.



If ∠X ≡∠M and $\frac{ZX}{PM} = \frac{XY}{MN}$ then ΔXYZ ≡ΔMNP.

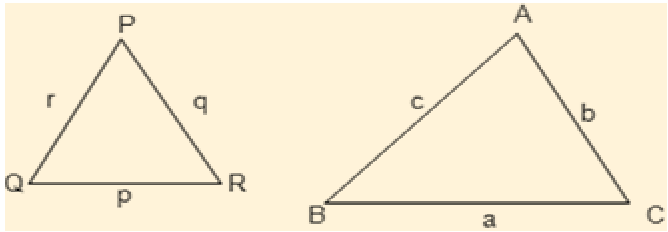
Proof: left for the reader

Theorem 3.4.5:

If the ratio of the lengths of the corresponding sides of two similar triangles is *k*, then

i. The ratio of their perimeters is *k*
ii. The ratio of their areas is

Given ΔABC andΔPQR



Proof: $\frac{\mathrm{AB}}{\mathrm{PQ}} = \frac{\mathrm{BC}}{\mathrm{QR}} = \frac{\mathrm{AC}}{\mathrm{PR}} \Longrightarrow \frac{c}{\mathrm{r}} = \frac{\mathrm{a}}{\mathrm{p}} = \frac{\mathrm{b}}{\mathrm{q}}$

Since the common value of these ratios is k, we have

$$\frac{c}{\mathrm{r}} = \frac{\mathrm{a}}{\mathrm{p}} = \frac{\mathrm{b}}{\mathrm{q}} = k. \text{ So, c} = k\text{r, a} = k\text{p, b} = k\text{q.}$$

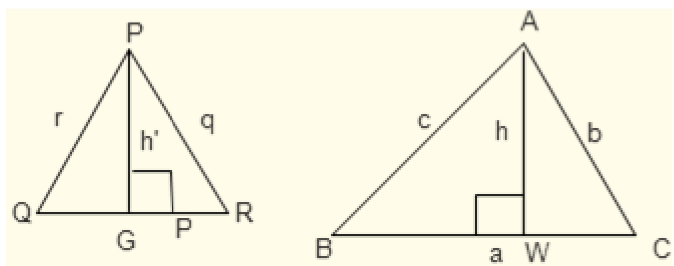
Now the perimeter of ΔABC = AB + BC + CA = c + a + b = kr+ kp+ kq.

From this, we obtain c + a + b = kn+ kl + km = k(r+ p+ q)

Therefore, $\frac{\text{perimeter of }\Delta\mathrm{ABC}}{\text{perimeter of }\Delta\mathrm{PQR}} = \frac{c+a+b}{\mathrm{r+p+q}} = \frac{k\,(\mathrm{r + p + q})}{\mathrm{r+p+q}} = k$

This shows that the ratio of their perimeters = the ratio of their corresponding sides.

ii, To prove that the ratio of their areas is the square of the ratio of any two corresponding sides:



Let ΔQPR≅ ΔBAC. Then,

$$\frac{\mathrm{AB}}{\mathrm{PQ}} = \frac{\mathrm{BC}}{\mathrm{QR}} = \frac{\mathrm{AC}}{\mathrm{PR}} \Longrightarrow \frac{c}{\mathrm{r}} = \frac{\mathrm{a}}{\mathrm{p}} = \frac{\mathrm{b}}{\mathrm{q}} = k$$

Let $\overline{\mathrm{PG}}$is an altitude from P to $\overline{\mathrm{QR}}$ and$\overline{\mathrm{AW}}$is an altitude from A to $\overline{\mathrm{BC}}$.

Since ΔQPG and ΔBAW are right triangles and ∠Q≡ ∠B,

We have ΔQPG ~ ΔBAW (AA similarity)

Therefore, $\frac{\mathrm{h}}{\mathrm{h}'} = \frac{\mathrm{a}}{\mathrm{p}} = k \Longrightarrow h = k\mathrm{h}', a = k\mathrm{p}$

Now, area of □□QPR = $\frac{1}{2}\mathrm{ph}'$, and area triangle BAC = $\frac{1}{2}\mathrm{ah}$

Therefore, $\frac{\text{Area of }\Delta\mathrm{ABC}}{\text{Area of }\Delta\mathrm{PQR}} = \frac{\frac{1}{2}\mathrm{ah}}{\frac{1}{2}\mathrm{ph}'} = \frac{\mathrm{ah}}{\mathrm{ph}'} = \frac{k\mathrm{p}k\mathrm{h}'}{\mathrm{ph}'} = k(k) = k^2$

So, the ratio of their areas is the square of the ratio of their corresponding sides.

Now we state the same fact for any two similar polygons.

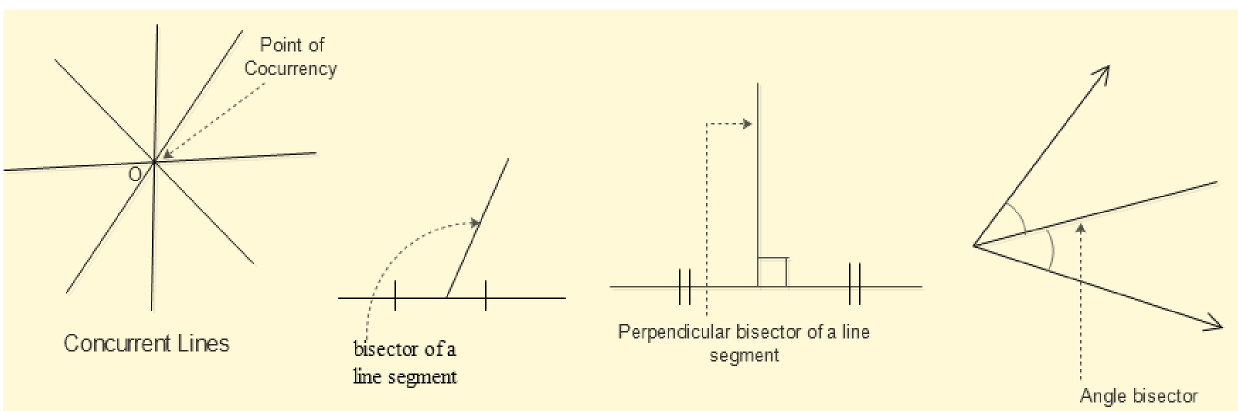
Activity:

1. Two triangles are similar. A side of one is 2 units long. The corresponding side of the other is 5 units long. What is the ratio of:
   a. Their perimeters? b. Their areas?
2. Two triangles are similar. The sides of one are three times as long as the sides of the other. What is the ratio of the areas of the smaller to the larger?

3.6. Some Theorems on Triangles

Theorems about collinear points and concurrent lines are called **incidence theorems**. Some such theorems are stated below.

Recall that a line that divides an angle into two congruent angles is called an **angle bisector** of the angle. A line that divides a line segment into two congruent line segments is called a **bisector** of the line segment. When a bisector of a line segment forms right angle with the line segment, then it is called the **perpendicular bisector** of the line segment. Three or more lines intersect at a common point, then the lines are called **concurrent lines** and their point of intersection is called the **point of concurrency**.



Perpendicular bisector of a line segment has the following special properties.

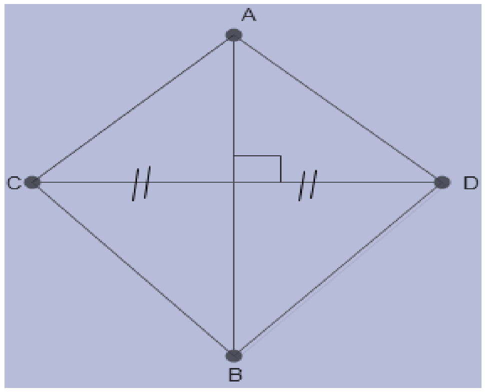
- A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

  *i.e.*If $\overline{AB} \perp \overline{CD}$ and $\overline{AB}$ bisects $\overline{CD}$ then $\overline{AC} = \overline{AD}$ and $\overline{BC} = \overline{BD}$

- Any point equidistant from the endpoints of a line segment lies on the perpendicular bisector of the segment.

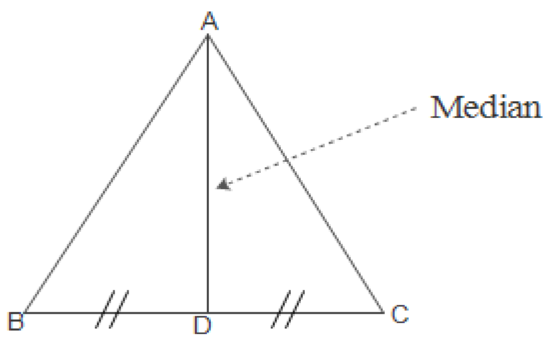
  i.e. If AC = AD, then A lies on the perpendicular bisector of$\overline{CD}$.

If BC = BD, then Blies on the perpendicular bisector of$\overline{CD}$.



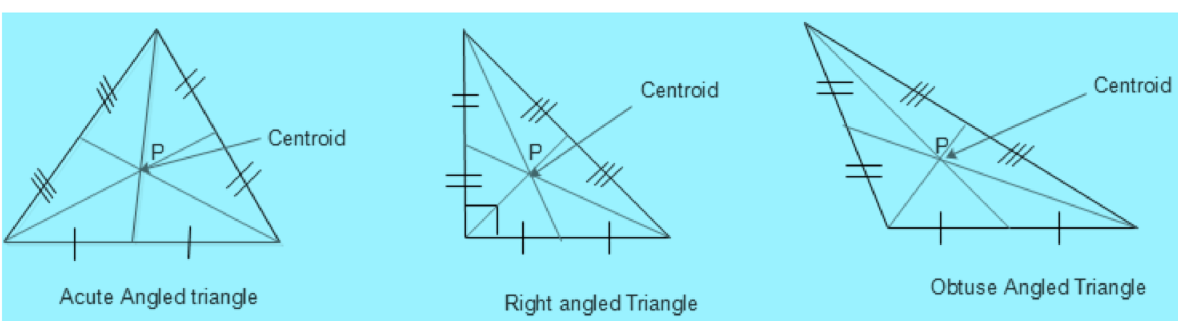
3.6.1 Median of a triangle

Median of a triangle is a line segment drawn from any vertex to the mid-point of the opposite side of a triangle. For instance, in ΔABCshown below, *D* is the midpoint of side BC. So, AD is a median of the triangle.



$\overline{AD}$is a median of triangle ABC.

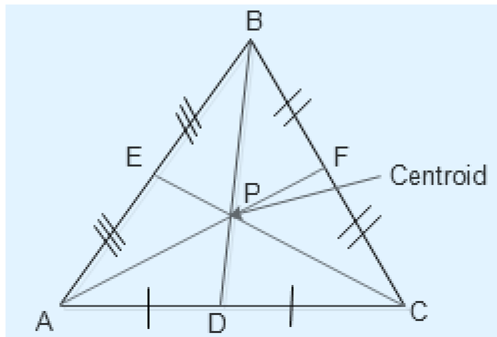
Any triangles are three medians. The three medians of a triangle are concurrent. The point of concurrency is called the **centroid of the triangle**. The centroid, labeled by point P in the diagrams below, is always inside the triangle. The centroid is the point of balance for any triangle.



The following theorem tells you that the three medians of a triangle intersect at one point. This point is called the **centroid** of the triangle.

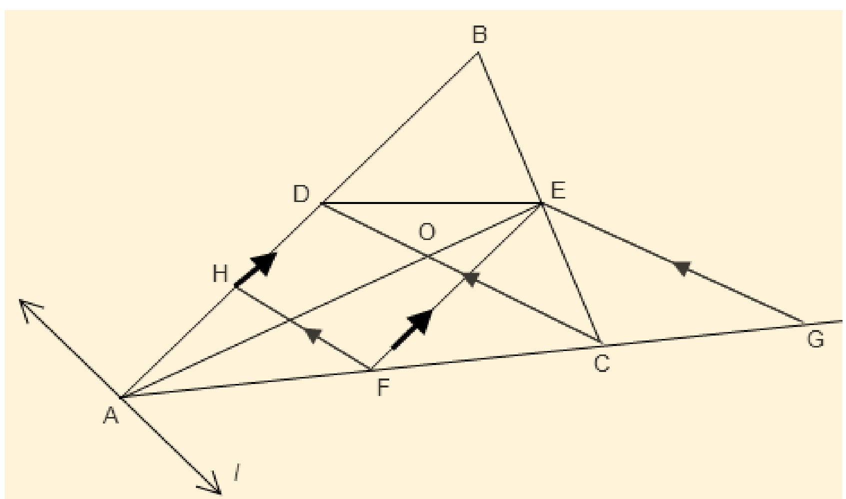
Theorem 3.6.1. Concurrency of Medians of a Triangle

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side. In the figure below;



If P is the centroid of ΔABC, then $AP = \frac{2}{3}AF$, $BP = \frac{2}{3}BD$ and $CP = \frac{2}{3}CE$

Proof: Suppose $\overline{AE}$ and $\overline{DC}$ are medians of ΔABC that are intersecting at point O. See Figure below



1. In ΔABC, $\overline{AE}$ and $\overline{DC}$ are medians intersecting at point O ------------Given
2. Draw $\overline{DE}$ --- Construction
3. Draw $\overline{EG}$ parallel to $\overline{DC}$ with G on the extension of $\overline{AC}$---------------- Construction
4. Draw $\overline{EF}$ parallel to $\overline{AB}$ with F on $\overline{AC}$ ---------------------------------- Construction
5. Draw $\overline{FH}$ parallel to $\overline{DC}$ with H on $\overline{AB}$ ----------------------------------- Construction
6. Draw line ℓ parallel to $\overline{DC}$ passing through A -------------------------- Construction
7. AFED and CGED are parallelograms with common side $\overline{DE}$ ----- Steps 3 and 4

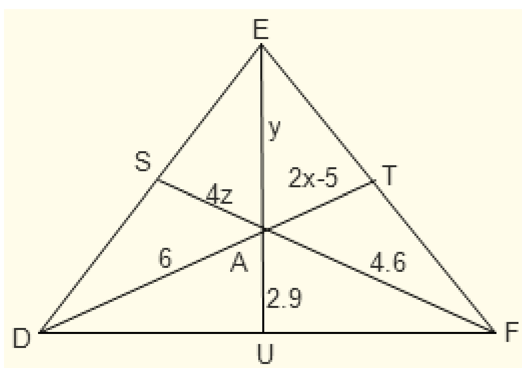
8. Therefore, AF = DE = CG -- Step 7
9. DE = $\frac{1}{2}$AC = AF --- Step 1,8 and side splitter Theorem
10. AF = FC = CG -- Steps 8 and 9
11. $\overline{AG}$is trisected by parallel lines *l*, $\overleftrightarrow{HF}$, $\overleftrightarrow{DC}$and $\overleftrightarrow{EG}$ ---------------- --- Steps 3, 5, 6 and 10
12. $\overline{AE}$is trisected by ℓ, $\overline{HF}$, $\overline{DC}$and $\overline{EG}$ ----------------------------------- Step 11 and property of parallel lines

Therefore, OE = $\frac{1}{3}$AE and AO = $\frac{2}{3}$AE

We have proved that the medians $\overline{DC}$and $\overline{AE}$meet at point O such that AO = $\frac{2}{3}$AE. Similarly; $\overline{AE}$ and $\overline{BF}$intersect at the same point O whose distance from *A* is$\frac{2}{3}$ of AE or from B is $\frac{2}{3}$ of BF.

Therefore, all the three medians of D*ABC* are concurrent at a single point *O* located at$\frac{2}{3}$of the distance from each vertex to the mid-point of the opposite side.

Example: In the figure below; Points S, T, and U are midpoints of $\overline{DE}$, $\overline{EF}$ and $\overline{DF}$ respectively. Find x, yand z.



Solution

Point A is point of concurrency of medians of a triangle, then

$$\mathrm{DA} = \frac{2}{3}\mathrm{DT}, \text{but } \mathrm{DA} = 6 \text{ and } \mathrm{DT} = 6 + 2\mathrm{x} - 5 = 2\mathrm{x} + 1, \text{then}$$

$$\mathrm{DA} = \frac{2}{3}\mathrm{DT} \Rightarrow 6 = \frac{2}{3}(2\mathrm{x} + 1)$$

$$\Rightarrow \frac{4}{3}\mathrm{x} = 6 - \frac{2}{3}$$

$$\Rightarrow \frac{4}{3}x = \frac{16}{3}$$

$$\Rightarrow x = \left(\frac{16}{3}\right)\left(\frac{3}{4}\right) = 4$$

$$UA = \frac{1}{3}UE, \qquad \text{but } UA = 2.9 \text{ and } UE = 2.9 + y, \qquad \text{then}$$

$$UA = \frac{1}{3}UE \Rightarrow 2.9 = \frac{1}{3}(2.9 + y)$$

$$\Rightarrow \frac{1}{3}y = \frac{29}{10} - \frac{29}{30}$$

$$\Rightarrow \frac{1}{3}y = \frac{58}{30}$$

$$\Rightarrow y = (\frac{58}{30})\,(3)$$

$$\Rightarrow y = \frac{58}{10} \Rightarrow y = 5.8$$

And

$$AF = \frac{2}{3}SF, \qquad \text{but } AF = 4.6 \text{ and } SF = 4.6 + 4z, \qquad \text{then}$$

$$AF = \frac{2}{3}SF \Rightarrow 4.6 = \frac{2}{3}(4.6 + 4z)$$

$$\Rightarrow (3)(4.6) = (2)(4.6) + 8z$$

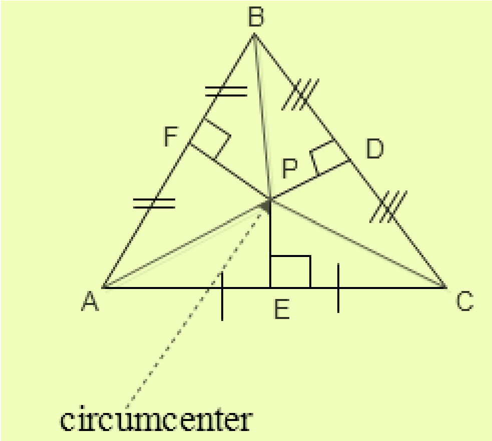
$$\Rightarrow 8z = 13.8 - 9.2$$

$$\Rightarrow 8z = 4.6$$

$$\Rightarrow z = \frac{4.6}{8} = \frac{46}{80} = 0.575$$

3.6.2 Perpendicular Bisector of a Side of a Triangle

A **perpendicular bisector** of side of a triangle is a line or a line segment or a ray in the same plane as the triangle that is perpendicular to the side and passes through its midpoint. The point of concurrency ofthe perpendicular bisectors of a triangle is called the **circumcenter**.



The circum center of a triangle is equidistant from the vertices of the triangle.

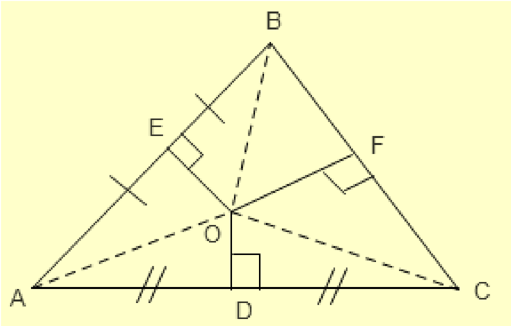
i. eIf P is the circum-center of ΔABC, then AP= BP= CP.

Theorem 3.6.2: Concurrency of Perpendicular bisectors of a Triangle

The perpendicular bisectors of sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

Proof: LetΔABC be given and construct perpendicular bisectors on any two of the sides. The perpendicular bisectors of $\overline{AB}$and $\overline{AC}$are shown in figure below. These perpendicular bisectors intersect at a point O; they cannot be parallel, since $\overline{AB}$and $\overline{AC}$ are sides of a triangle, so they are not parallel.

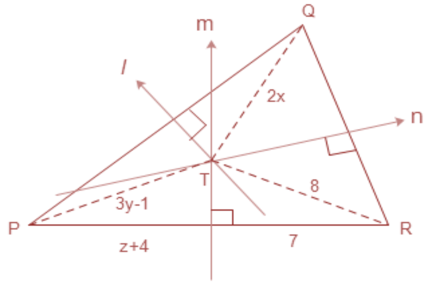
Note that the perpendicular bisector of the remaining side BC must pass through the point O. Therefore, the point of intersection of the three perpendicular bisectors is equidistant from the three vertices of ΔABC.



O is the point where the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$meet, as shown in the figure above, □AOD ≡ □COD by SAS and hence AO = CO. Similarly;□AOE ≡ □BOE by SAS and hence AO =BO, Thus, AO = BO = CO, It follows that O is equidistant from the vertices of □ABC.

Next, let F be the foot of the perpendicular from O to $\overline{BC}$. Then, OF is the perpendicular bisector of $\overline{BC}$because □BOC is an isosceles triangle. Therefore, the perpendicular bisectors of the sides of □ABC are concurrent.

Example: Lines *l*, *m*, and *n* are perpendicular bisectors of ΔPQR and meet at T. If TQ = 2x, PT = 3y -1, and TR = 8, find x, y, and z.



Solution: Exercise

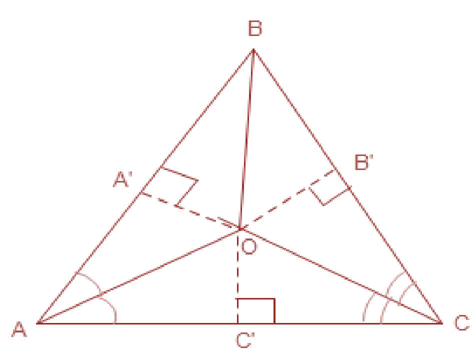
3.6.3 Angle Bisector of a Triangle;

An **angle bisector** of a triangle is a straight line or a line segment or a ray that divides an angle of a triangle in to two congruent angles. There are three angle bisectors in any triangle. The angle bisectors of a triangle are concurrent, and their point of concurrency is called the **incenter**of a triangle.

Theorem 3.6.3.Concurrency of Angle Bisectors of a Triangle

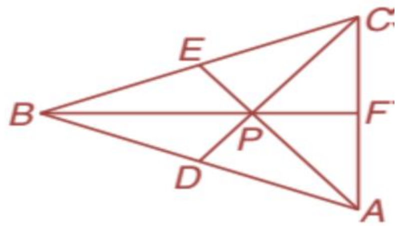
The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle. That means, If K is the incenter of ΔABC, then KP = KQ = KR. Where point P is a point on AB, point Q is A point on BC and point R is a point on AC.

Proof: To show that the angle bisectors of ΔABC meet at a single point, draw the bisectors of ∠A and ∠C, intersecting each other at O as seen in the figure below,



Construct the perpendiculars$\overline{OA'}$, $\overline{OB'}$and $\overline{OC'}$. Do these segments have the same length? Show that ΔOBB' ≡ΔOBA' by AAS and conclude that ∠OBB' ≡ ∠OBA'. This implies that the bisector of ∠B also passes through the point O. and therefore, the angle bisectors of ΔABC meet at a single point. Also their point of intersection is equidistant from the three sides of ΔABC.

Example: In ΔABC given below, $\overline{BF}$ is the angle bisector of ∠ABC, $\overline{BF}$, $\overline{AE}$ and $\overline{DC}$are medians, and P is the centroid.



a. Find x if DP = 4x - 3 and CP = 30.
b. Find y if AP = y and EP = 18.
c. Find z if FP = 5z - 10 and BP = 42.
d. If m(∠ABC) = x and m(∠BAC) = m(∠BCA) = 2x – 10 is $\overline{BF}$an altitude? Explain.

Solution: exercise

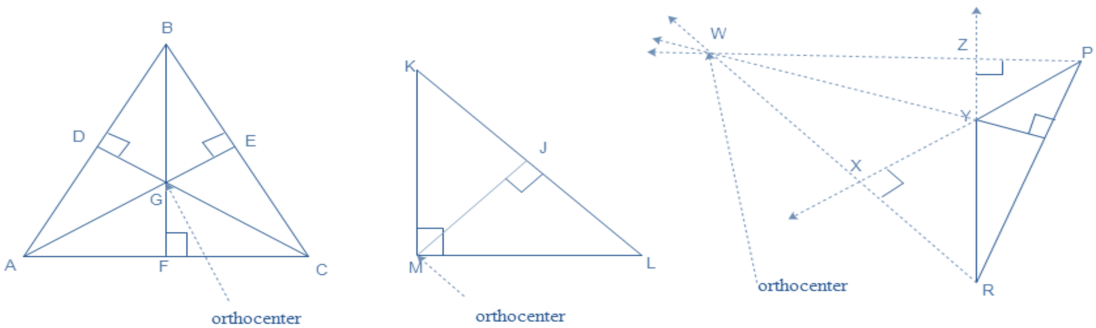
Activity:

1. For what kind of triangle is the medians and angle bisectors are the same segments?
2. For what kind of triangle is the centroid outside the triangle?

3.6.4 Altitude of a triangle

An altitude of a triangle is a line segment through a vertex perpendicular to a line containing the base(opposite side to the vertex). The line containing the opposite side of the vertex is called the **extended base of the altitude**. The intersection between the extended base and the altitude is called the **foot of the altitude.** The length of the altitude is simply the distance between the extended base and the vertex. Any triangle has three altitudes and drowns these altitudes by using any side as the base of their corresponding vertex. The lines containing the altitudes are concurrent and intersect at a point called **the orthocenter** of the triangle

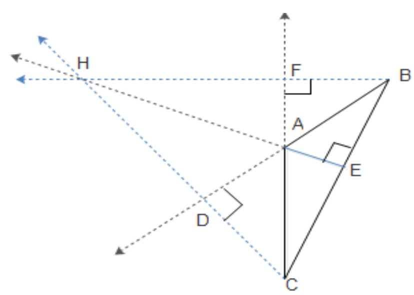
Note: An altitude may lie inside, on or outside of a triangle.



- $\Delta ABC$is an acute-angled triangle. The three altitudes intersect at G, a point **inside** the triangle. This implies that the orthocenter is inside of a triangle.
- $\Delta KLM$is a right triangle. The two legs, $\overline{LM}$and$\overline{KM}$, are also altitudes. They intersect at the triangle's right angle. This implies that the orthocenter is **on** the triangle at M, the vertex of the right angle of the triangle.
- $\Delta YPR$is an obtuse triangle. The three lines that contain the altitudes intersect at W, a point that is **outside** the triangle. This implies that the orthocenter is outside of a triangle.

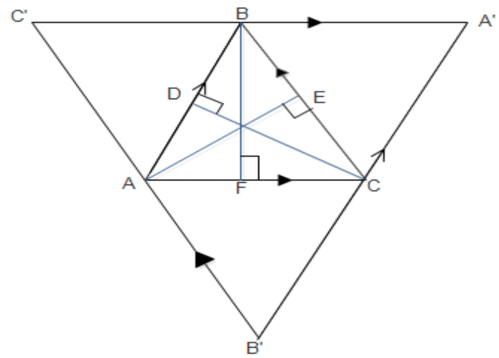
Theorem 3.5.4.Concurrency of Altitudes of a Triangle

The lines containing the altitudes of a triangle are concurrent.



i.e, If$\overline{AE}$, $\overline{BF}$and $\overline{CD}$ are the altitudes of ΔABC, then the lines$\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$ and $\overleftrightarrow{CD}$ intersect at some point H.

Proof: To show that the three altitudes of ΔABCmeet at a single point, construct $\Delta A'B'C'$as shown in the Figure below, so that the three sides of$\Delta A'B'C'$ are parallel respectively to the three sides ofΔABC.

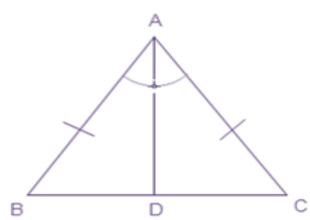


Let $\overline{EA}$, $\overline{BF}$and $\overline{CD}$are the altitudes of ΔABC. The quadrilaterals ABA'C, ABCB' and AC'BC are parallelograms (pair of opposite sides are parallel). Since ABA'C is a parallelogram, AC = BA' (Opposite sides of a parallelogram are congruent). Since ACBC' is a parallelogram, AC = BC'. Therefore, BC' = BA' (transitive property of a line segment) and BF bisects A'C'.

Accordingly, BF is perpendicular to AC and so BF is the perpendicular bisector of A'C'. Similarly, one can show that CD and AE are perpendicular bisectors of A'B' and B'C' respectively. Therefore, the altitudes of ΔABC are the same as the perpendicular bisectors of the sides of ΔA'B'C'. Since the perpendicular bisectors of any triangle are concurrent (theorem 3.3), it is therefore, true that the altitudes are concurrent.

Theorem 3.6.5. Isosceles Triangle Theorem

Angles opposite to equal sides of an isosceles triangle are congruent.



Proof: We are given an isosceles triangle ABC in which AB = AC.

We need to prove that = ∠C.

Let us draw the bisector of ∠A and D is point of intersection of this bisector and BC.

Now let us consider Δ BAD and ΔCAD.

Statement Reason

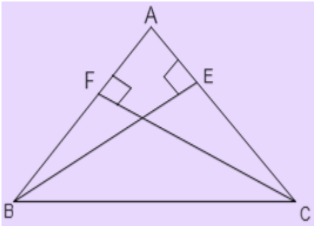
i. AB = AC -- Given
ii. ∠BAD = ∠CAD ------------------------------------ By construction
iii. AD = AD -- Common

So, $\Delta BAD \equiv \Delta CAD$, By SAS congruency theorem

$\angle ABD \equiv \angle ACD$, corresponding angles of congruent triangles

$\angle B \equiv \angle C$

Example:ΔABC is an isosceles triangle in which $\overline{CF}$ and $\overline{BE}$ are altitudes as shown in the figure below. Show that altitudes are equal.

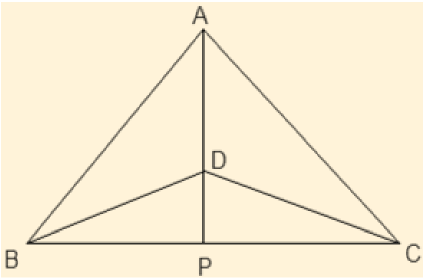


Proof: left for the reader

Activity: Is the converse of isosceles triangle theorem also true? That is: If two angles of any triangle are equal, can we conclude that the sides opposite to them are also equal?

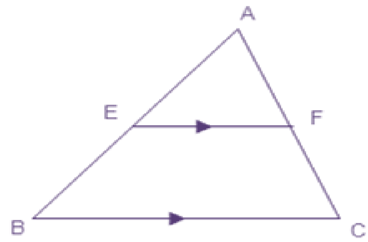
Exercise: ΔABC and ΔDBC are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC see in the figure given below. If AD is extended to intersect BC at P, show that

i. $\Delta ABD \equiv \Delta ACD$
ii. $\Delta ABP \equiv \Delta ACP$
iii. AP bisects $\angle A$ as well as $\angle D$.
iv. AP is the perpendicular bisector of BC



Theorem 3.6.6. Side Splitter Theorem

A line is parallel to one side of a triangle and intersects the other two sides, and then it divides those sides proportionally.



Given; $\overline{EF} \parallel \overline{BC}$

Prove: $\frac{AB}{AE} = \frac{AC}{AF}$

Solution: consider ΔAEF and ΔABC.

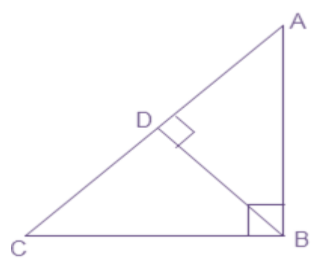
i. ∠A ≡∠A common angle

ii. ∠AEF ≡∠ABC corresponding angles

iii. ΔAEF ~ ΔABC, by AA similarity theorem

iv. $\frac{AB}{AE} = \frac{AC}{AF}$ Corresponding sides of similar Δ

Theorem3.6.7. An Altitude is drown to the hypotenuse of a right triangle.



a. The two newly formed right triangles are similar to the given right triangle and similar to each other.

   ΔADB ~ ΔABC

   ΔBDC ~ ΔABC

   ΔADB ~ ΔBDC

b. **(Altitude Theorem).** The length of the altitude is proportional to the length of segments on the hypotenuse. (i.e the length of an altitude to the hypotenuse square is equal to the product of the length of line segments on the hypotenuse formed by this altitude.)

   - $AD: BD = BD: CD \Leftrightarrow (BD)^2 = AD \,.\, CD$

c. **(Euiclud's Theorem).** The length of each leg of a given triangle is proportional between the lengths of the whole hypotenuse and the length of the projection of the leg on the hypotenuse.

   - $AC: AB = AB: AD \Leftrightarrow (AB)^2 = AC.\, AD$

- $AC : BC = BC : CD \Leftrightarrow (BC)^2 = AC . CD$

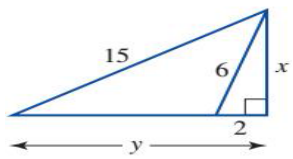
d. **(Pythagoras' Theorem)** For any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides."

$(AB)^2 + (BC)^2 = (AC)^2$

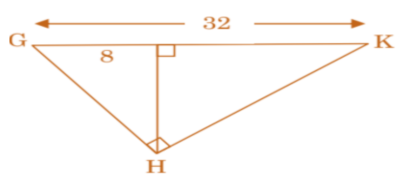
Proof: Exercise

Exercise

1. A right-angled triangle has a base of 5 cm and a perpendicular height of 11 cm. Find the length of the hypotenuse.
2. An isosceles triangle has a base of 30 cm and a perpendicular height of 10 cm. Find the length of the other two equal sides of the isosceles triangle.
3. The diagonal of a rectangle is 120 cm. One side has a length of 70 cm. Find: the length of the other side.
4. A ladder that is 7m long leans up against a vertical wall. The top of the ladder reaches 6.5 m up the wall. How far from the wall is the foot of the ladder?
5. For the diagram given below, find the length of the sides marked *x* and *y*.



6. The altitude of a right triangle divides the hypotenuse into two segments whose lengths are 9 cm and 16cm. Find the altitude and lengths of the two legs.
7. For the diagram given below**,** find the lengths of GH and HK.



8. When Ali planted a tree 5 m away from point A, the tree just blocked the view of a building 50 m away. If the building was 20 m tall, how tall was the tree?
9. A tree casts a shadow of 30 m. At the same time, a 10 m pole casts a shadow of12 m. Find the height of the tree.